

A NOTE ON FAILURE OF ENERGY REVERSAL FOR CLASSICAL FRACTIONAL SINGULAR INTEGRALS

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ABSTRACT. For $0 \leq \alpha < n$ we demonstrate the failure of energy reversal for the vector of α -fractional Riesz transforms, and more generally for the vector of all α -fractional convolution singular integrals having a kernel with vanishing integral on every great circle of the sphere.

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1. INTRODUCTION

To set notation we recall a special case of Theorem 1 from our paper [SaShUr], using notation from that paper.

Theorem 1. *Suppose that σ and ω are locally finite positive Borel measures in \mathbb{R}^n with no common point masses, and assume the finiteness of the α -energy condition constant*

$$(\mathcal{E}_\alpha)^2 \equiv \sup_{\substack{Q=\cup Q_r \\ Q, Q_r \in \mathcal{D}^n}} \frac{1}{|I|_\sigma} \sum_{r=1}^{\infty} \sum_{J \in \mathcal{M}_{\mathbf{r}-\text{deep}}(Q_r)} \left(\frac{P^\alpha(J, \mathbf{1}_Q \sigma)}{|J|^{\frac{1}{n}}} \right)^2 \left\| \mathbf{P}_J^{\text{subgood}, \omega} \mathbf{x} \right\|_{L^2(\omega)}^2 \\ + \sup_{\ell \geq 0} \frac{1}{|I|_\sigma} \sum_{J \in \mathcal{M}_{\mathbf{r}-\text{deep}}^\ell(Q)} \left(\frac{P^\alpha(J, \mathbf{1}_Q \sigma)}{|J|^{\frac{1}{n}}} \right)^2 \left\| \mathbf{P}_J^{\text{subgood}, \omega} \mathbf{x} \right\|_{L^2(\omega)}^2,$$

and its dual, uniformly over all dyadic grids \mathcal{D}^n , and where the goodness parameters \mathbf{r} and ε implicit in the definition of $\mathcal{M}_{\text{deep}}(K)$ are fixed sufficiently large and small respectively depending on n and α . Let \mathbf{T}^α be a standard strongly elliptic α -fractional Calderón-Zygmund operator in Euclidean space \mathbb{R}^n . Then \mathbf{T}^α is bounded from $L^2(\sigma)$ to $L^2(\omega)$ if and only if the \mathcal{A}_2^α condition

$$(1.1) \quad \mathcal{A}_2^\alpha \equiv \sup_{Q \in \mathcal{Q}^n} \mathcal{P}^\alpha(Q, \sigma) \frac{|Q|_\omega}{|Q|^{1-\frac{\alpha}{n}}} < \infty$$

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and its dual hold, the cube testing conditions

$$(1.2) \quad \int_Q |\mathbf{T}^\alpha(\mathbf{1}_Q \sigma)|^2 \omega \leq \mathfrak{T}_{T^\alpha}^2 \int_Q d\sigma \text{ and } \int_Q |(\mathbf{T}^\alpha)^*(\mathbf{1}_Q \omega)|^2 \sigma \leq \mathfrak{T}_{T^\alpha}^2 \int_Q d\omega,$$

hold for all cubes Q in \mathbb{R}^n , and the weak boundedness property for \mathbf{T}^α holds:

$$\left| \int_Q \mathbf{T}^\alpha(\mathbf{1}_{Q'} \sigma) d\omega \right| \leq \mathcal{WB}\mathcal{P}_{\mathbf{T}^\alpha} \sqrt{|Q|_\omega |Q'|_\sigma},$$

for all cubes Q, Q' with $\frac{1}{C} \leq \frac{|Q|^{\frac{1}{n}}}{|Q'|^{\frac{1}{n}}} \leq C$,

and either $Q \subset 3Q' \setminus Q'$ or $Q' \subset 3Q \setminus Q$.

In [SaShUr3] we used Theorem 1 to prove the $T1$ theorem for the vector of Riesz transforms in \mathbb{R}^n in the special case when one of the measures σ, ω is supported on a line in \mathbb{R}^n . The key to that proof was proving control of the above energy constants \mathcal{E}_α and \mathcal{E}_α^* in terms of the constants in the hypotheses (1.1) and (1.2). A number of attempts have been made by us and others (see e.g. earlier versions of [SaShUr] and [LaWi]) to prove such control of various different energy conditions by invoking an *energy reversal* for the Riesz transforms and similar operators - see (2.4) below - but all of these attempts have been met with failure. The purpose of this short note is to show first that *energy reversal* is false, not only for the vector of α -fractional Riesz transforms in the plane when $0 \leq \alpha < 2$, but also for the vectors of classical α -fractional singular integrals in the plane,

$$\begin{aligned} \mathbf{T}_M^\alpha &\equiv \{T_\Omega : \Omega \in \mathcal{P}_M\}, \\ \mathcal{P}_M &\equiv \{\cos n\theta, \sin n\theta\}_{n=1}^M, \end{aligned}$$

where T_Ω^α has convolution kernel $\frac{\Omega(\frac{x}{|x|})}{|x|^{2-\alpha}} = \frac{\Omega(\theta)}{|x|^{2-\alpha}}$ and $0 \leq \alpha < 2$. The linear space \mathcal{L}_M of trigonometric polynomials with vanishing mean and degree at most M is spanned by the monomials \mathcal{P}_M , and so we also obtain the failure of energy reversal for the infinite vector $\mathbf{T}_M^\alpha \equiv \{T_\Omega : \Omega \in \mathcal{L}_M\}$. A standard limiting argument applied to the proof below extends this failure to all sufficiently smooth $\Omega(\theta)$ with vanishing mean on the circle. Finally, we embed an analogue of the planar measure constructed below into Euclidean space \mathbb{R}^n in order to obtain the failure of energy reversal for *any* vector of classical convolution Calderón-Zygmund operators with odd kernel in \mathbb{R}^n - and more generally for kernels $\frac{\Omega(x')}{|x|^{n-\alpha}}$ where Ω has vanishing integral on every great circle in the sphere \mathbb{S}^{n-1} . A key to our proof is the positivity of the determinants $\det \left[\frac{\Gamma(z)^2}{\Gamma(z-|i-j|)\Gamma(z+|i-j|)} \right]_{i,j=1}^n$ for all $n \geq 1$. See also [LaWi] for related results regarding fractional Riesz transforms in higher dimensions. We thank Michael Lacey for pointing out to us that the 1-fractional Riesz transform $\mathbf{R}^1 \sigma(z) = \int_{\mathbb{T}} \frac{z-\xi}{|z-\xi|^2} d\sigma(\xi)$ of the unit circle measure σ vanishes identically for z inside the unit disk. Indeed, $\mathbf{R}^1 \sigma$ is the gradient of the planar Newtonian potential $\mathbf{N}\sigma(z) = \int_{\mathbb{T}} \ln|z-\xi| d\sigma(\xi)$, and $\mathbf{N}\sigma$ is constant inside the disk.

2. FAILURE OF REVERSAL OF ENERGY

Recall the energy $E(J, \omega)$ of ω on a cube J ,

$$E(J, \omega)^2 \equiv \frac{1}{|J|_\omega} \frac{1}{|J|_\omega} \int_J \int_J \left| \frac{x-z}{|J|^{\frac{1}{n}}} \right|^2 d\omega(x) d\omega(z) = 2 \frac{1}{|J|_\omega} \int_J \left| \frac{x - \mathbb{E}_J^\omega x}{|J|^{\frac{1}{n}}} \right|^2 d\omega(x).$$

Define its associated *coordinate* energies $E^j(J, \omega)$ by

$$E^j(J, \omega)^2 \equiv \frac{1}{|J|_\omega} \frac{1}{|J|_\omega} \int_J \int_J \left| \frac{x^j - z^j}{|J|^{\frac{1}{n}}} \right|^2 d\omega(x) d\omega(z), \quad j = 1, 2, \dots, n,$$

and the rotations $E_{\mathcal{R}}^j(J, \omega)$ of the coordinate energies by a rotation $\mathcal{R} \in SO(n)$, which we refer to as *partial* energies,

$$E_{\mathcal{R}}^j(J, \omega)^2 \equiv \frac{1}{|J|_\omega} \frac{1}{|J|_\omega} \int_J \int_J \left| \frac{x_{\mathcal{R}}^j - z_{\mathcal{R}}^j}{|J|^{\frac{1}{n}}} \right|^2 d\omega(x) d\omega(z), \quad j = 1, 2, \dots, n,$$

where for $\mathcal{R} \in SO(n)$, $x_{\mathcal{R}} = (x_{\mathcal{R}}^j)_{j=1}^n = \mathcal{R}(x^j)_{j=1}^n = \mathcal{R}x$. Set $E_{\mathcal{R}}(J, \omega)^2 \equiv E_{\mathcal{R}}^1(J, \omega)^2 + \dots + E_{\mathcal{R}}^n(J, \omega)^2$. We have the following elementary computations.

Lemma 1. *For $\mathcal{R} \in SO(n)$ we have*

$$(2.1) \quad E_{\mathcal{R}}(J, \omega)^2 = E_{\mathcal{R}}^1(J, \omega)^2 + \dots + E_{\mathcal{R}}^n(J, \omega)^2 = E(J, \omega)^2.$$

More generally, if $\mathfrak{R} = \{\mathcal{R}_j\}_{j=1}^n \subset SO(n)$ is a collection of rotations such that the

matrix $M_{\mathfrak{R}} = \begin{bmatrix} \mathcal{R}_1 \mathbf{e}^1 \\ \vdots \\ \mathcal{R}_n \mathbf{e}^1 \end{bmatrix}$ with rows $\mathcal{R}_\ell \mathbf{e}^1$ is nonsingular, then

$$(2.2) \quad E(J, \omega)^2 \leq \frac{1}{\epsilon_{\mathfrak{R}}} \sum_{\ell=1}^n E_{\mathcal{R}_\ell}^1(J, \omega)^2,$$

where $\epsilon_{\mathfrak{R}}$ is the least eigenvalue of $M_{\mathfrak{R}}^* M_{\mathfrak{R}}$.

Proof. We have

$$\begin{aligned} |x_{\mathcal{R}}^1 - z_{\mathcal{R}}^1|^2 + \dots + |x_{\mathcal{R}}^n - z_{\mathcal{R}}^n|^2 &= |\mathcal{R}(x - z)|^2 \\ &= |x - z|^2 = |x^1 - z^1|^2 + \dots + |x^n - z^n|^2, \end{aligned}$$

so that

$$\begin{aligned} E_{\mathcal{R}}(J, \omega)^2 &\equiv E_{\mathcal{R}}^1(J, \omega)^2 + \dots + E_{\mathcal{R}}^n(J, \omega)^2 \\ &= E^1(J, \omega)^2 + \dots + E^n(J, \omega)^2 = E(J, \omega)^2. \end{aligned}$$

More generally, if $M_{\mathfrak{R}}^\ell$ denotes the ℓ^{th} row of the matrix $M_{\mathfrak{R}}$, we have

$$\begin{aligned} \epsilon_{\mathfrak{R}} |x - z|^2 &\leq (x - z)^{tr} M_{\mathfrak{R}}^* M_{\mathfrak{R}} (x - z) \\ &= \sum_{\ell=1}^n |\mathcal{R}_\ell \mathbf{e}^1 \cdot (x - z)|^2, \end{aligned}$$

so that

$$\begin{aligned}
\epsilon_{\Re} \mathbf{E}(J, \omega)^2 &= \left(\frac{1}{|J|_{\omega} |J|^{\frac{1}{n}}}} \right)^2 \int_J \int_J \epsilon_{\Re} |x - z|^2 d\omega(x) d\omega(z) \\
&\leq \left(\frac{1}{|J|_{\omega} |J|^{\frac{1}{n}}}} \right)^2 \int_J \int_J \left\{ \sum_{\ell=1}^n |\mathcal{R}_{\ell} \mathbf{e}^1 \cdot (x - z)|^2 \right\} d\omega(x) d\omega(z) \\
&= \sum_{\ell=1}^n \mathbf{E}_{\mathcal{R}_{\ell}}^1(J, \omega)^2.
\end{aligned}$$

□

The point of the estimate (2.2) is that it could hopefully be used to help obtain a reversal of energy for a vector transform $\mathbf{T}^{n, \alpha} = \{T_{\ell}^{n, \alpha}\}_{\ell=1}^n$, where the convolution kernel $K_{\ell}^{n, \alpha}(w)$ of the operator $T_{\ell}^{n, \alpha}$ has the form

$$(2.3) \quad K_{\ell}^{n, \alpha}(w) = \frac{\Omega_{\ell}^n \left(\frac{w}{|w|} \right)}{|w|^{n-\alpha}},$$

and where Ω_{ℓ}^n is smooth on the sphere \mathbb{S}^{n-1} . We refer to the operator $T_{\ell}^{n, \alpha}$ as an α -fractional *convolution* Calderón-Zygmund operator. If in addition we require that Ω_{ℓ}^n has vanishing integral on the sphere \mathbb{S}^{n-1} , we refer to $T_{\ell}^{n, \alpha}$ as a *classical* α -fractional Calderón-Zygmund operator.

However, we now dash this hope, at least for the most familiar singular operators in the plane, in a spectacular way. A vector $\mathbf{T}^{\alpha} = \{T_{\ell}^{\alpha}\}_{\ell=1}^N$ of α -fractional transforms in Euclidean space \mathbb{R}^n satisfies a *strong* reversal of ω -energy on a cube J if there is a positive constant C_0 such that for all $\gamma \geq 2$ sufficiently large and for all positive measures μ supported outside γJ , we have the inequality

$$(2.4) \quad \mathbf{E}(J, \omega)^2 \mathbf{P}^{\alpha}(J, \mu)^2 \leq C_0 \mathbb{E}_J^{d\omega(x)} \mathbb{E}_J^{d\omega(z)} |\mathbf{T}^{\alpha} \mu(x) - \mathbf{T}^{\alpha} \mu(z)|^2.$$

We show that (2.4) is false by stating and proving a variant of Lemma 9 in [SaShUr2].

Lemma 2 (Failure of Reverse Energy). *Suppose that J is a square in the plane \mathbb{R}^2 , $0 \leq \alpha < 2$, $\gamma > 2$ and that $\mathbf{R}^{\alpha} = \{R_{\ell}^{\alpha}\}_{\ell=1}^2$ is the vector of α -fractional Riesz transforms in the plane \mathbb{R}^2 with kernels $K_{\ell}^{\alpha}(w) = \frac{\Omega_{\ell}(\frac{w}{|w|})}{|w|^{2-\alpha}}$ and $\Omega_{\ell}(\frac{w}{|w|}) = \frac{w_{\ell}}{|w|}$. Finally suppose that $C_0 > 0$ is given. For γ sufficiently large, there exists a positive measure μ on \mathbb{R}^2 supported outside γJ and depending only on α and γ , such that the strong reversal of energy inequality (2.4) **fails**. Moreover, we can choose μ as above so that in addition, for any $M \geq 1$, the strong reversal of energy inequality (2.4) fails for the vector \mathbf{T}_M^{α} .*

As a corollary of the proof of this lemma we easily obtain an extension to higher dimensions by simply embedding an appropriate planar measure into Euclidean space \mathbb{R}^n .

Corollary 1 (of the proof of Lemma 2). *Suppose that J is a cube in \mathbb{R}^n , $0 \leq \alpha < n$, $\gamma > 2$ and suppose that $C_0 > 0$ is given. For γ sufficiently large, there exists a positive measure μ on \mathbb{R}^n supported outside γJ and depending only on n , α and γ , such that the strong reversal of energy inequality (2.4) **fails** for any vector*

$\mathbf{T}^\alpha = \{T_\ell^\alpha\}_{\ell=1}^N$ of α -fractional smooth Calderón-Zygmund operators in \mathbb{R}^n with kernels $K_\ell^\alpha(w) = \frac{\Omega_\ell(\frac{w}{|w|})}{|w|^{n-\alpha}}$, where Ω_ℓ has vanishing integral on every great circle in the sphere \mathbb{S}^{n-1} - in particular this holds if each K_ℓ^α is odd.

Proof of Lemma 2 for the Riesz transform vector. Let $\varepsilon > 0$. We let $\frac{\Omega_\ell(\frac{w}{|w|})}{|w|^{2-\alpha}}$ be an arbitrary standard kernel for the moment. With $K_\ell^\alpha(x, y) = K_\ell^\alpha(x - y)$ we have

$$\begin{aligned} T_\ell^\alpha \mu(x) &= \int K_\ell^\alpha(x - y) d\mu(y) = \int \frac{\Omega_\ell(x - y)}{|y - x|^{2-\alpha}} d\mu(y) \\ &= \int \{K_\ell^\alpha(c_J - y) + (x - c_J) \cdot \nabla K_\ell^\alpha(c_J - y)\} d\mu(y) + E_{\ell,x}, \end{aligned}$$

and so

$$\begin{aligned} &T_\ell^\alpha \mu(x) - T_\ell^\alpha \mu(z) \\ &= \int \{(x - z) \cdot \nabla K_\ell^\alpha(c_J - y)\} d\mu(y) + [E_{\ell,x}^\alpha - E_{\ell,z}^\alpha] \\ &\equiv \Lambda_\ell^\alpha + [E_{\ell,x}^\alpha - E_{\ell,z}^\alpha], \end{aligned}$$

where if $\gamma > 2$ is sufficiently large,

$$(2.5) \quad |E_{\ell,x}^\alpha - E_{\ell,z}^\alpha| \leq C \frac{1}{\gamma^\delta} \frac{P^\alpha(J, \mu)}{|J|^{\frac{1}{2}}} |x - z| \leq \varepsilon \frac{P^\alpha(J, \mu)}{|J|^{\frac{1}{2}}} |x - z|.$$

The point of this inequality (2.5) is that it permits the replacement of the difference $T_\ell^\alpha \mu(x) - T_\ell^\alpha \mu(z)$ in (2.4) by the linear part Λ_ℓ^α of the Taylor expansion of the kernel K_ℓ^α .

Now we make the choice

$$\begin{aligned} \Omega_\ell(w) &= \Omega(\theta_\ell(w)); \\ \theta_\ell(w) &\equiv \tan^{-1} \frac{(-1)^{\ell'} w^{\ell'}}{w^\ell}, \quad 1 \leq \ell \leq 2, \end{aligned}$$

where $w^{\ell'}$ denotes the coordinate variable other than w^ℓ , i.e. $\ell + \ell' = 3$. Thus θ_1 is the usual angular coordinate on the circle and $\theta_2 = \theta_1 + \frac{\pi}{2}$. We now use

$$\begin{aligned} \nabla |w|^{\alpha-2} &= \left(\frac{\partial}{\partial w^1} \left((w^1)^2 + (w^2)^2 \right)^{\frac{\alpha-2}{2}}, \frac{\partial}{\partial w^2} \left((w^1)^2 + (w^2)^2 \right)^{\frac{\alpha-2}{2}} \right) \\ &= \frac{\alpha-2}{2} \left((w^1)^2 + (w^2)^2 \right)^{\frac{\alpha-2}{2}-1} 2w \\ &= (\alpha-2) |w|^{\alpha-4} w. \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial w^\ell} \tan^{-1} \frac{w^{\ell'}}{w^\ell} &= \frac{1}{1 + \left(\frac{w^{\ell'}}{w^\ell} \right)^2} \frac{-w^{\ell'}}{(w^\ell)^2} = \frac{-w^{\ell'}}{|w|^2}, \\ \frac{\partial}{\partial w^{\ell'}} \tan^{-1} \frac{w^{\ell'}}{w^\ell} &= \frac{1}{1 + \left(\frac{w^{\ell'}}{w^\ell} \right)^2} \frac{1}{w^\ell} = \frac{w^\ell}{|w|^2}. \end{aligned}$$

to calculate that the gradient of the convolution kernel

$$K_\ell^\alpha(w) = \frac{\Omega_\ell(w)}{|w|^{2-\alpha}} = \frac{\Omega(\theta_\ell(w))}{|w|^{2-\alpha}} = \frac{\Omega\left(\tan^{-1} \frac{w^{\ell'}}{w^\ell}\right)}{|w|^{2-\alpha}},$$

is given by,

$$\begin{aligned} \nabla K_\ell^\alpha(w) &= \nabla \left(\frac{\Omega_\ell(w)}{|w|^{2-\alpha}} \right) = \Omega(\theta_\ell(w)) \nabla |w|^{\alpha-2} + |w|^{\alpha-2} \Omega'(\theta_\ell(w)) \nabla \theta_\ell \\ &= \frac{(\alpha-2) \Omega(\theta_\ell(w)) w + \Omega'(\theta_\ell(w)) w^\perp}{|w|^{4-\alpha}}. \end{aligned}$$

Thus the linear part Λ_ℓ^α in the Taylor expansion of $T_\ell^\alpha \mu$ is given by

$$\Lambda_\ell^\alpha = (x-z) \cdot \int \nabla K_\ell^\alpha(c_J - y) d\mu(y) \equiv (x-z) \cdot \mathbf{Z}_{\Omega_\ell}^\alpha(c_J; \mu),$$

where

$$\begin{aligned} \mathbf{Z}_{\Omega_\ell}^\alpha(c_J; \mu) &= \int_{\mathbb{R}^2} \frac{(\alpha-2) \Omega(\theta_\ell(c_J - y)) (c_J - y) + \Omega'(\theta_\ell(c_J - y)) (c_J - y)^\perp}{|c_J - y|^{4-\alpha}} d\mu(y) \\ &= \int_{w \in \mathbb{S}^1} \{(\alpha-2) \Omega(\theta_\ell(w)) w^1 - \Omega'(\theta_\ell(w)) w^2\} \mathbf{e}^1 d\Psi_\mu(w) \\ &\quad + \int_{w \in \mathbb{S}^1} \{(\alpha-2) \Omega(\theta_\ell(w)) w^2 + \Omega'(\theta_\ell(w)) w^1\} \mathbf{e}^2 d\Psi_\mu(w), \end{aligned}$$

and \mathbf{e}^ℓ is the coordinate vector with a 1 in the ℓ^{th} position. Here the measure Ψ_μ is an essentially *arbitrary* positive finite measure on the circle \mathbb{S}^1 given formally by

$$d\Psi_\mu(w) = \int_0^\infty r^{\alpha-3} d\mu_w(r) = \int_0^\infty r^{\alpha-3} d\mu(rw), \quad w \in \mathbb{S}^1.$$

We use,

$$\begin{aligned} \tan \theta_\ell(w) &= \frac{(-1)^{\ell'} w^{\ell'}}{w^\ell}, \\ \csc \theta_\ell(w) &= (-1)^{\ell'} \sqrt{1 + \cot^2 \theta_\ell(w)} = (-1)^{\ell'} \sqrt{1 + \left(\frac{w^\ell}{w^{\ell'}}\right)^2} = \frac{|w|}{(-1)^{\ell'} w^{\ell'}}, \\ \sin \theta_\ell(w) &= \frac{(-1)^{\ell'} w^{\ell'}}{|w|} \text{ and } \cos \theta_\ell(w) = \frac{w^\ell}{|w|}, \end{aligned}$$

for $w \neq 0$, to obtain

$$\begin{aligned} \mathbf{Z}_{\Omega_1}^\alpha(c_J; \mu) &= \int_{\mathbb{S}^1} \{(\alpha-2) \Omega(\theta_1(w)) \cos \theta_1(w) - \Omega'(\theta_1(w)) \sin \theta_1(w)\} \mathbf{e}^1 d\Psi_\mu \\ &\quad + \int_{\mathbb{S}^1} \int \{(\alpha-2) \Omega(\theta_1(w)) \sin \theta_1(w) + \Omega'(\theta_1(w)) \cos \theta_1(w)\} \mathbf{e}^2 d\Psi_\mu \\ &\equiv \int_{\mathbb{S}^1} \{A_\alpha^1(\theta_1(w)) \mathbf{e}^1 + B_\alpha^1(\theta_1(w)) \mathbf{e}^2\} d\Psi_\mu, \end{aligned}$$

and

$$\begin{aligned} \mathbf{Z}_{\Omega_2}^\alpha(c_J; \mu) &= \int_{\mathbb{S}^1} \{-(\alpha-2)\Omega(\theta_2(w))\sin\theta_2(w) - \Omega'(\theta_2(w))\cos\theta_2(w)\} \mathbf{e}^1 d\Psi_\mu \\ &\quad + \int_{\mathbb{S}^1} \{(\alpha-2)\Omega(\theta_2(w))\cos\theta_2(w) - \Omega'(\theta_2(w))\sin\theta_2(w)\} \mathbf{e}^2 d\Psi_\mu \\ &\equiv \int_{\mathbb{S}^1} \{A_\alpha^2(\theta_2(w))\mathbf{e}^1 + B_\alpha^2(\theta_2(w))\mathbf{e}^2\} d\Psi_\mu, \end{aligned}$$

with

$$\begin{aligned} (2.6) \quad A_\alpha^1(t) &= (\alpha-2)\Omega(t)\cos t - \Omega'(t)\sin t = B_\alpha^2(t), \\ B_\alpha^1(t) &= (\alpha-2)\Omega(t)\sin t + \Omega'(t)\cos t = -A_\alpha^2(t). \end{aligned}$$

Now we show below in (2.11) that a necessary condition for reversal of energy on J is that the span of the pair of vectors $\{\mathbf{Z}_{\Omega_\ell}^\alpha(c_J; \mu)\}_{\ell=1}^2$ is all of \mathbb{R}^2 :

$$(2.7) \quad \text{Span}\{\mathbf{Z}_{\Omega_\ell}^\alpha(c_J; \mu)\}_{\ell=1}^2 = \mathbb{R}^2.$$

So it suffices to show the failure of (2.7), i.e. that $\mathbf{Z}_{\Omega_1}^\alpha(c_J; \mu)$ and $\mathbf{Z}_{\Omega_2}^\alpha(c_J; \mu)$ are parallel.

At this point we take $\ell = 1$ and set $\theta = \theta_1(w)$ so that we obtain

$$\begin{aligned} (2.8) \quad A_\alpha(\theta) &\equiv A_\alpha^1(\theta_1(w)) = (\alpha-2)\Omega(\theta)\cos\theta - \Omega'(\theta)\sin\theta, \\ B_\alpha(\theta) &\equiv B_\alpha^1(\theta_1(w)) = (\alpha-2)\Omega(\theta)\sin\theta + \Omega'(\theta)\cos\theta. \end{aligned}$$

In the case $\alpha = 1$ these coefficients are perfect derivatives,

$$\begin{aligned} A_1(\theta) &= -\Omega(\theta)\cos\theta - \Omega'(\theta)\sin\theta = -[\Omega(\theta)\sin\theta]', \\ B_1(\theta) &= -\Omega(\theta)\sin\theta + \Omega'(\theta)\cos\theta = -[\Omega(\theta)\cos\theta]', \end{aligned}$$

and so have vanishing integral on the circle. Thus with the choice $d\Psi_\mu(\theta) = d\theta$ we have

$$\mathbf{Z}_\Omega(c_J; \mu) = \int_{\mathbb{S}^1} \{A_1(\theta)\mathbf{e}^1 + B_1(\theta)\mathbf{e}^2\} d\theta = \mathbf{0}$$

the zero vector, for **every** choice of differentiable Ω on the circle.

In the case $0 \leq \alpha < 2$ with $\alpha \neq 1$, it is no longer possible to find a nontrivial measure μ so that $\mathbf{Z}_\Omega^\alpha(c_J; \mu)$ vanishes for all differentiable Ω , but we will see that we *can* always find a positive measure μ such that the vectors $\mathbf{Z}_{\Omega_1}^\alpha(c_J; \mu)$ and $\mathbf{Z}_{\Omega_2}^\alpha(c_J; \mu)$ are parallel for the choice $\Omega(\theta) = \cos\theta$ that corresponds to the vector of Riesz transforms.

Indeed, in the special case that $\Omega(t) = \cos t$, and recalling that $\theta_2(w) = \theta_1(w) + \frac{\pi}{2} = \theta + \frac{\pi}{2}$, we have

$$\begin{aligned} A_\alpha^1(\theta_1(w)) &= A_\alpha^1(\theta) = (\alpha-2)\cos^2\theta + \sin^2\theta; \\ B_\alpha^1(\theta_1(w)) &= B_\alpha^1(\theta) = (\alpha-3)\cos\theta\sin\theta; \\ A_\alpha^2(\theta_2(w)) &= -B_\alpha^1\left(\theta + \frac{\pi}{2}\right) = -(\alpha-3)\cos\left(\theta + \frac{\pi}{2}\right)\sin\left(\theta + \frac{\pi}{2}\right) \\ &= (\alpha-3)\cos\theta\sin\theta; \\ B_\alpha^2(\theta_2(w)) &= A_\alpha^1\left(\theta + \frac{\pi}{2}\right) = (\alpha-2)\cos^2\left(\theta + \frac{\pi}{2}\right) + \sin^2\left(\theta + \frac{\pi}{2}\right) \\ &= (\alpha-2)\sin^2\theta + \cos^2\theta. \end{aligned}$$

Thus we also have

$$\begin{aligned}
\mathbf{Z}_{\Omega_1}^\alpha(c_J; \mu) &= \int_{\mathbb{S}^1} \{A_\alpha^1(\theta_1(w)) \mathbf{e}^1 + B_\alpha^1(\theta_1(w)) \mathbf{e}^2\} d\Psi_\mu \\
&= \int_{\mathbb{S}^1} \{[(\alpha-2)\cos^2\theta + \sin^2\theta] \mathbf{e}^1 + [(\alpha-3)\cos\theta\sin\theta] \mathbf{e}^2\} d\Psi_\mu \\
&= \left\{ \int_{\mathbb{S}^1} [(\alpha-2)\cos^2\theta + \sin^2\theta] d\Psi_\mu \right\} \mathbf{e}^1 + \left\{ \int_{\mathbb{S}^1} [(\alpha-3)\cos\theta\sin\theta] d\Psi_\mu \right\} \mathbf{e}^2
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{Z}_{\Omega_2}^\alpha(c_J; \mu) &= \int_{\mathbb{S}^1} \{A_\alpha^2(\theta_2(w)) \mathbf{e}^1 + B_\alpha^2(\theta_2(w)) \mathbf{e}^2\} d\Psi_\mu \\
&= \int_{\mathbb{S}^1} \{[(\alpha-3)\cos\theta\sin\theta] \mathbf{e}^1 + [(\alpha-2)\sin^2\theta + \cos^2\theta] \mathbf{e}^2\} d\Psi_\mu \\
&= \left\{ \int_{\mathbb{S}^1} [(\alpha-3)\cos\theta\sin\theta] d\Psi_\mu \right\} \mathbf{e}^1 + \left\{ \int_{\mathbb{S}^1} [(\alpha-2)\sin^2\theta + \cos^2\theta] d\Psi_\mu \right\} \mathbf{e}^2.
\end{aligned}$$

Using

$$\begin{aligned}
(2.9) \quad &(\alpha-2)\cos^2\theta + \sin^2\theta = (\alpha-3)\cos^2\theta + 1, \\
&(\alpha-2)\sin^2\theta + \cos^2\theta = (\alpha-3)\sin^2\theta + 1, \\
&\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta, \quad \cos^2\theta = \frac{1+\cos 2\theta}{2}, \quad \sin^2\theta = \frac{1-\cos 2\theta}{2},
\end{aligned}$$

we see that

$$\begin{aligned}
(\alpha-2)\cos^2\theta + \sin^2\theta &= (\alpha-3)\frac{1+\cos 2\theta}{2} + 1 = \frac{\alpha-3}{2}\cos 2\theta + \frac{\alpha-1}{2}, \\
(\alpha-2)\sin^2\theta + \cos^2\theta &= (\alpha-3)\frac{1-\cos 2\theta}{2} + 1 = -\frac{\alpha-3}{2}\cos 2\theta + \frac{\alpha-1}{2}, \\
(\alpha-3)\cos\theta\sin\theta &= \frac{\alpha-3}{2}\sin 2\theta.
\end{aligned}$$

Plugging these formulas into those for $\mathbf{Z}_{\Omega_1}^\alpha(c_J; \mu)$ and $\mathbf{Z}_{\Omega_2}^\alpha(c_J; \mu)$ we obtain

$$\begin{aligned}
&\det \begin{bmatrix} \mathbf{Z}_{\Omega_1}^\alpha(c_J; \mu) \\ \mathbf{Z}_{\Omega_2}^\alpha(c_J; \mu) \end{bmatrix} \\
&= \det \begin{bmatrix} \int_{\mathbb{S}^1} \left[\frac{\alpha-3}{2}\cos 2\theta + \frac{\alpha-1}{2} \right] d\Psi_\mu & \int_{\mathbb{S}^1} \left[\frac{\alpha-3}{2}\sin 2\theta \right] d\Psi_\mu \\ \int_{\mathbb{S}^1} \left[\frac{\alpha-3}{2}\sin 2\theta \right] d\Psi_\mu & \int_{\mathbb{S}^1} \left[-\frac{\alpha-3}{2}\cos 2\theta + \frac{\alpha-1}{2} \right] d\Psi_\mu \end{bmatrix} \\
&= \left(\frac{\alpha-3}{2} \int_{\mathbb{S}^1} \cos 2\theta d\Psi_\mu + \frac{\alpha-1}{2} \|\Psi_\mu\| \right) \left(-\frac{\alpha-3}{2} \int_{\mathbb{S}^1} \cos 2\theta d\Psi_\mu + \frac{\alpha-1}{2} \|\Psi_\mu\| \right) \\
&\quad - \left(\frac{\alpha-3}{2} \int_{\mathbb{S}^1} \sin 2\theta d\Psi_\mu \right)^2 \\
&= \left(\frac{\alpha-1}{2} \|\Psi_\mu\| \right)^2 - \left\{ \left(\frac{\alpha-3}{2} \int_{\mathbb{S}^1} \cos 2\theta d\Psi_\mu \right)^2 + \left(\frac{\alpha-3}{2} \int_{\mathbb{S}^1} \sin 2\theta d\Psi_\mu \right)^2 \right\}.
\end{aligned}$$

Thus $\det \begin{bmatrix} \mathbf{Z}_{\Omega_1}^\alpha(c_J; \mu) \\ \mathbf{Z}_{\Omega_2}^\alpha(c_J; \mu) \end{bmatrix} = 0$ if and only if the length of the vector

$$\frac{\alpha-3}{2} \begin{pmatrix} \int_{\mathbb{S}^1} \cos 2\theta d\Psi_\mu \\ \int_{\mathbb{S}^1} \sin 2\theta d\Psi_\mu \end{pmatrix}$$

equals $\frac{|\alpha-1|}{2} \|\Psi_\mu\|$, i.e.

$$(2.10) \quad \left\| \begin{pmatrix} \int_{\mathbb{S}^1} \cos 2\theta d\Psi_\mu \\ \int_{\mathbb{S}^1} \sin 2\theta d\Psi_\mu \end{pmatrix} \right\| = \frac{|\alpha-1|}{|\alpha-3|} \|\Psi_\mu\|.$$

To construct a positive probability measure $d\Psi_\mu$ on the circle that satisfies (2.10), we first observe that if $d\Psi_\mu = \delta_0$ is the unit point mass at 0, then

$$\left\| \begin{pmatrix} \int_{\mathbb{S}^1} \cos 2\theta d\Psi_\mu \\ \int_{\mathbb{S}^1} \sin 2\theta d\Psi_\mu \end{pmatrix} \right\| = \left\| \begin{pmatrix} \int_{\mathbb{S}^1} d\Psi_\mu \\ 0 \end{pmatrix} \right\| = \|\Psi_\mu\|,$$

and since $|\alpha-1| < |\alpha-3|$ for all $0 \leq \alpha < 2$, we have

$$\left\| \begin{pmatrix} \int_{\mathbb{S}^1} \cos 2\theta d\Psi_\mu \\ \int_{\mathbb{S}^1} \sin 2\theta d\Psi_\mu \end{pmatrix} \right\| > \frac{|\alpha-1|}{|\alpha-3|} \|\Psi_\mu\|,$$

in this case. On the other hand, if $d\Psi_\mu(\theta) = \frac{1}{2\pi}d\theta$ is normalized Lebesgue measure on the circle, we have

$$\left\| \begin{pmatrix} \int_{\mathbb{S}^1} \cos 2\theta d\Psi_\mu \\ \int_{\mathbb{S}^1} \sin 2\theta d\Psi_\mu \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\| = 0 < \frac{|\alpha-1|}{|\alpha-3|} \|\Psi_\mu\|.$$

It is now easy to see that there is a convex combination $d\Psi_\mu = (1-\lambda)\delta_0 + \lambda\frac{1}{2\pi}d\theta$ such that (2.10) holds. Thus (2.7) fails, and we now show that energy reversal fails.

In fact, we may assume that both $\mathbf{Z}_{\Omega_1}^\alpha(c_J; \mu)$ and $\mathbf{Z}_{\Omega_2}^\alpha(c_J; \mu)$ are parallel to the coordinate vector \mathbf{e}_2 , and in this case we will see that we can reverse at most the coordinate energy $\mathbf{E}^2(J, \omega)$, defined above by

$$\mathbf{E}^2(J, \omega)^2 \equiv \frac{1}{|J|_\omega} \frac{1}{|J|_\omega} \int_J \int_J \left| \frac{x^2 - z^2}{|J|^{\frac{1}{n}}} \right|^2 d\omega(x) d\omega(z),$$

and not the full energy $\mathbf{E}(J, \omega)$. More precisely, we claim that there is a measure ω such that for γ so large that $\varepsilon \ll C_0$, the strong reversal of ω -energy inequality (2.4) fails. Indeed, using that $\mathbf{Z}_{\Omega_\ell}^\alpha(c_J)(c_J)$ is parallel to \mathbf{e}^2 , we have that

$$(2.11) \quad \begin{aligned} & \int_J \int_J |\mathbf{T}^\alpha \mu(x) - \mathbf{T}^\alpha \mu(z)|^2 d\omega(x) d\omega(z) \\ &= \sum_{\ell=1}^2 \int_J \int_J |(x-z) \cdot \mathbf{Z}_{\Omega_\ell}^\alpha(c_J) + [E_{\ell,x}^\alpha - E_{\ell,z}^\alpha]|^2 d\omega(x) d\omega(z) \\ &\leq \sum_{\ell=1}^2 \int_J \int_J \left| \frac{\mathbf{P}^\alpha(J, \mu)}{|J|^{\frac{1}{2}}} (x-z) \cdot \frac{\mathbf{Z}_{\Omega_\ell}^\alpha(c_J)(c_J)}{|\mathbf{Z}_{\Omega_\ell}^\alpha(c_J)(c_J)|} \right|^2 d\omega(x) d\omega(z) \\ &\quad + C \sum_{\ell=1}^2 \int_J \int_J \left| \varepsilon \frac{\mathbf{P}^\alpha(J, \mu)}{|J|^{\frac{1}{2}}} |x-z| \right|^2 d\omega(x) d\omega(z) \\ &\leq \mathbf{E}^2(J, \omega)^2 \mathbf{P}^\alpha(J, \mu)^2 + C\varepsilon^2 \mathbf{E}(J, \omega)^2 \mathbf{P}^\alpha(J, \mu)^2 \\ &\leq \frac{1}{10} C_0 \mathbf{E}(J, \omega)^2 \mathbf{P}^\alpha(J, \mu)^2, \end{aligned}$$

provided we choose γ so large that $C\varepsilon^2 \leq \frac{1}{10}C_0$ and provided we choose ω so that $\mathbf{E}^2(J, \omega) = 0$ but $\mathbf{E}(J, \omega) > 0$. This completes the proof of the first assertion in Lemma 2. \square

Remark 1. *The condition (2.10) must be invariant under rotations, i.e. invariant under replacing θ by $\theta - \phi$ for any constant ϕ , and this is easily seen using (2.9) above:*

$$\begin{aligned} \begin{pmatrix} \int_{\mathbb{S}^1} \cos 2(\theta - \phi) d\Psi_\mu \\ \int_{\mathbb{S}^1} \sin 2(\theta - \phi) d\Psi_\mu \end{pmatrix} &= \begin{pmatrix} \cos 2\phi \int_{\mathbb{S}^1} \cos 2\theta d\Psi_\mu + \sin 2\phi \int_{\mathbb{S}^1} \sin 2\theta d\Psi_\mu \\ \cos 2\phi \int_{\mathbb{S}^1} \sin 2\theta d\Psi_\mu - \sin 2\phi \int_{\mathbb{S}^1} \cos 2\theta d\Psi_\mu \end{pmatrix} \\ &= \cos 2\phi \begin{pmatrix} \int_{\mathbb{S}^1} \cos 2\theta d\Psi_\mu \\ \int_{\mathbb{S}^1} \sin 2\theta d\Psi_\mu \end{pmatrix} - \sin 2\phi \begin{pmatrix} \int_{\mathbb{S}^1} \cos 2\theta d\Psi_\mu \\ \int_{\mathbb{S}^1} \sin 2\theta d\Psi_\mu \end{pmatrix}^\perp, \end{aligned}$$

which has length independent of ϕ .

Remark 2. *The above proof shows that for each $t \in \mathbb{R}$, the convolution kernel*

$$\Phi_{\alpha,t}(x,y) = \frac{x \cos t + y \sin t}{(x^2 + y^2)^{\frac{3-\alpha}{2}}},$$

in the plane with coordinates (x,y) , $x,y \in \mathbb{R}$, and the probability measure $d\mu_\alpha$ supported on the circle $\mathbb{S}^1 = [0, 2\pi)$ given by

$$d\mu_\alpha(\theta) = \frac{|\alpha - 1|}{|\alpha - 3|} \delta_0(\theta) + \frac{|\alpha - 3| - |\alpha - 1|}{|\alpha - 3|} \frac{d\theta}{2\pi},$$

satisfy the property that $\text{grad}(\Phi_{\alpha,t} * \mu_\alpha)(0,0)$ points in the same direction for all t . A direct calculation shows that

$$\text{grad}(\Phi_{\alpha,t} * \mu_\alpha)(0,0) = (\alpha - 1) \begin{cases} [\cos t, 0] & \text{for } 0 \leq \alpha < 1 \\ [0, \sin t] & \text{for } 1 < \alpha < 2 \end{cases}.$$

Indeed, if for $\theta \in \mathbb{R}$ we define $\Phi_{\alpha,t}^\theta$ to be the convolution of $\Phi_{\alpha,t}$ with the unit point mass $\delta_{e^{i\theta}}$ at $e^{i\theta}$ in the circle,

$$\Phi_{\alpha,t}^\theta(x,y) \equiv (\Phi_{\alpha,t} * \delta_{e^{i\theta}})(x,y) = \frac{(x - \cos \theta) \cos t + (y - \sin \theta) \sin t}{\left((x - \cos \theta)^2 + (y - \sin \theta)^2\right)^{\frac{3-\alpha}{2}}},$$

then we have

$$\begin{aligned} \text{grad} \Phi_{\alpha,t}^\theta(x,y) &= \left[\left(\frac{\partial}{\partial x} \Phi_{\alpha,t}^\theta \right)(x,y), \left(\frac{\partial}{\partial y} \Phi_{\alpha,t}^\theta \right)(x,y) \right] \\ &= \frac{[\cos t, \sin t]}{\left((x - \cos \theta)^2 + (y - \sin \theta)^2\right)^{\frac{3-\alpha}{2}}} \\ &\quad - \frac{3-\alpha}{2} \{(x - \cos \theta) \cos t + (y - \sin \theta) \sin t\} \frac{[2(x - \cos \theta), 2(y - \sin \theta)]}{\left((x - \cos \theta)^2 + (y - \sin \theta)^2\right)^{\frac{5-\alpha}{2}}}, \end{aligned}$$

and when $(x,y) = (0,0)$ we get

$$\text{grad} \Phi_{\alpha,t}^\theta(0,0) = [\cos t, \sin t] - (3-\alpha) \{\cos \theta \cos t + \sin \theta \sin t\} [\cos \theta, \sin \theta].$$

Thus we have

$$\begin{aligned} \text{grad} \Phi_t^0(0,0) &= \text{grad} \Phi_t^0(0,0) = [\cos t, \sin t] - (3-\alpha) \cos t [1, 0] \\ &= [-(2-\alpha) \cos t, \sin t], \end{aligned}$$

and

$$\begin{aligned}
\text{grad} \left(\Phi_t * \frac{d\theta}{2\pi} \right) (0, 0) &= \text{grad} \int_0^{2\pi} (\Phi_t * \delta_{e^{i\theta}}) (0, 0) \frac{d\theta}{2\pi} \\
&= [\cos t, \sin t] - \frac{3-\alpha}{2} [\cos t, \sin t] \\
&= \left[\frac{\alpha-1}{2} \cos t, \frac{\alpha-1}{2} \sin t \right].
\end{aligned}$$

Thus

$$\begin{aligned}
&(3-\alpha) \text{grad} (\Phi_{\alpha,t} * \mu_\alpha) (0, 0) \\
&= |\alpha-1| [-(2-\alpha) \cos t, \sin t] + (|\alpha-3| - |\alpha-1|) \left[\frac{\alpha-1}{2} \cos t, \frac{\alpha-1}{2} \sin t \right] \\
&= \left[\left\{ -(2-\alpha) |\alpha-1| + (|\alpha-3| - |\alpha-1|) \frac{\alpha-1}{2} \right\} \cos t, \left\{ |\alpha-1| + (|\alpha-3| - |\alpha-1|) \frac{\alpha-1}{2} \right\} \sin t \right].
\end{aligned}$$

Now for $0 \leq \alpha < 1$ we get

$$|\alpha-1| + (|\alpha-3| - |\alpha-1|) \frac{\alpha-1}{2} = 1 - \alpha + 2 \frac{\alpha-1}{2} = 0,$$

and

$$-(2-\alpha) |\alpha-1| + (|\alpha-3| - |\alpha-1|) \frac{\alpha-1}{2} = (\alpha-1) (3-\alpha).$$

For $1 < \alpha < 2$ we get

$$|\alpha-1| + (|\alpha-3| - |\alpha-1|) \frac{\alpha-1}{2} = (\alpha-1) (3-\alpha),$$

and

$$-(2-\alpha) |\alpha-1| + (|\alpha-3| - |\alpha-1|) \frac{\alpha-1}{2} = 0.$$

Proof of Lemma 2 for the vector of trig polynomials. Recall that with $\theta = \theta_1(w)$ we obtain

$$\begin{aligned}
A_\alpha(\theta) &= (\alpha-2) \Omega(\theta) \cos \theta - \Omega'(\theta) \sin \theta \\
B_\alpha(\theta) &= (\alpha-2) \Omega(\theta) \sin \theta + \Omega'(\theta) \cos \theta.
\end{aligned}$$

Thus we have

$$\begin{aligned}
A_\alpha(\theta) &= \{(\alpha-2) \Omega(\theta) + i\Omega'(\theta)\} \{\cos \theta + i \sin \theta\} \\
&\quad - i \{(\alpha-2) \Omega(\theta) \sin \theta + \Omega'(\theta) \cos \theta\} \\
&= \{(\alpha-2) \Omega(\theta) + i\Omega'(\theta)\} \{\cos \theta + i \sin \theta\} - i B_\alpha(\theta),
\end{aligned}$$

and so

$$\{(\alpha-2) \Omega(\theta) + i\Omega'(\theta)\} \{\cos \theta + i \sin \theta\} = A_\alpha(\theta) + i B_\alpha(\theta).$$

This shows that in complex notation,

$$\begin{aligned}
\mathbf{Z}_\Omega^\alpha(c_J; \mu) &= \int_{\mathbb{S}^1} \{A_\alpha(\theta) + i B_\alpha(\theta)\} d\Psi_\mu \\
&= \int_{\mathbb{S}^1} \{(\alpha-2) \Omega(\theta) + i\Omega'(\theta)\} \{\cos \theta + i \sin \theta\} d\Psi_\mu \\
&= \int_{\mathbb{S}^1} \Omega_\alpha(\theta) e^{i\theta} d\Psi_\mu,
\end{aligned}$$

where

$$\Omega_\alpha(\theta) \equiv (\alpha - 2)\Omega(\theta) + i\Omega'(\theta).$$

Recall the product formulas

$$\begin{aligned} 2 \cos A \cos B &= \cos(A - B) + \cos(A + B); \\ 2 \sin A \sin B &= \cos(A - B) - \cos(A + B); \\ 2 \sin A \cos B &= \sin(A - B) + \sin(A + B). \end{aligned}$$

In the special case that $\Omega_1^k(t) = \cos kt$ we thus have

$$\begin{aligned} A_\alpha(\theta) &= (\alpha - 2) \cos k\theta \cos \theta + k \sin k\theta \sin \theta \\ &= (\alpha - 2) \frac{1}{2} [\cos(k - 1)\theta + \cos(k + 1)\theta] \\ &\quad + k \frac{1}{2} [\cos(k - 1)\theta - \cos(k + 1)\theta] \\ &= \left\{ \frac{\alpha + k}{2} - 1 \right\} \cos(k - 1)\theta + \left\{ \frac{\alpha - k}{2} - 1 \right\} \cos(k + 1)\theta; \\ B_\alpha(\theta) &= (\alpha - 2) \cos k\theta \sin \theta - k \sin k\theta \cos \theta \\ &= (\alpha - 2) \frac{1}{2} [-\sin(k - 1)\theta + \sin(k + 1)\theta] \\ &\quad - k \frac{1}{2} [\sin(k - 1)\theta + \sin(k + 1)\theta] \\ &= -\left\{ \frac{\alpha + k}{2} - 1 \right\} \sin(k - 1)\theta + \left\{ \frac{\alpha - k}{2} - 1 \right\} \sin(k + 1)\theta, \end{aligned}$$

and so

$$\begin{aligned} \mathbf{Z}_{\Omega_1^k}^\alpha(c_J; \mu) &= \int_{\mathbb{S}^1} \{A_\alpha(\theta) \mathbf{e}^1 + B_\alpha(\theta) \mathbf{e}^2\} d\Psi_\mu \\ &= \int_{\mathbb{S}^1} \left[\left\{ \frac{\alpha + k}{2} - 1 \right\} \cos(k - 1)\theta + \left\{ \frac{\alpha - k}{2} - 1 \right\} \cos(k + 1)\theta \right] d\Psi_\mu \mathbf{e}^1 \\ &\quad + \int_{\mathbb{S}^1} \left[-\left\{ \frac{\alpha + k}{2} - 1 \right\} \sin(k - 1)\theta + \left\{ \frac{\alpha - k}{2} - 1 \right\} \sin(k + 1)\theta \right] d\Psi_\mu \mathbf{e}^2 \\ &= \left\{ \frac{\alpha + k - 2}{2} \right\} \int_{\mathbb{S}^1} \begin{pmatrix} \cos(k - 1)\theta \\ -\sin(k - 1)\theta \end{pmatrix} d\Psi_\mu \\ &\quad + \left\{ \frac{\alpha - k - 2}{2} \right\} \int_{\mathbb{S}^1} \begin{pmatrix} \cos(k + 1)\theta \\ \sin(k + 1)\theta \end{pmatrix} d\Psi_\mu \\ &= \int_{\mathbb{S}^1} \left\{ \left(\frac{\alpha + k - 2}{2} \right) e^{-i(k - 1)\theta} + \left(\frac{\alpha - k - 2}{2} \right) e^{i(k + 1)\theta} \right\} d\Psi_\mu \\ &= \left(\frac{\alpha + k - 2}{2} \right) \widehat{\Psi}_\mu(k - 1) + \left(\frac{\alpha - k - 2}{2} \right) \widehat{\Psi}_\mu(k + 1). \end{aligned}$$

Next we take $\Omega_2^k(\theta) = \sin k\theta$ so that

$$\begin{aligned}
A_\alpha(\theta) &= (\alpha - 2) \sin k\theta \cos \theta - k \cos k\theta \sin \theta \\
&= (\alpha - 2) \frac{1}{2} [\sin(k-1)\theta + \sin(k+1)\theta] \\
&\quad - k \frac{1}{2} [-\sin(k-1)\theta + \sin(k+1)\theta] \\
&= \left\{ \frac{\alpha+k}{2} - 1 \right\} \sin(k-1)\theta + \left\{ \frac{\alpha-k}{2} - 1 \right\} \sin(k+1)\theta; \\
B_\alpha(\theta) &= (\alpha - 2) \sin k\theta \sin \theta + k \cos k\theta \cos \theta \\
&= (\alpha - 2) \frac{1}{2} [\cos(k-1)\theta - \cos(k+1)\theta] \\
&\quad + k \frac{1}{2} [\cos(k-1)\theta + \cos(k+1)\theta] \\
&= \left\{ \frac{\alpha+k}{2} - 1 \right\} \cos(k-1)\theta - \left\{ \frac{\alpha-k}{2} - 1 \right\} \cos(k+1)\theta.
\end{aligned}$$

Thus with $\Omega_2^k(\theta) = \sin k\theta$ we obtain

$$\begin{aligned}
\mathbf{Z}_{\Omega_2^k}^\alpha(c_J; \mu) &= \int_{\mathbb{S}^1} \{A_\alpha(\theta) \mathbf{e}^1 + B_\alpha(\theta) \mathbf{e}^2\} d\Psi_\mu \\
&= \int_{\mathbb{S}^1} \left[\left\{ \frac{\alpha+k}{2} - 1 \right\} \sin(k-1)\theta + \left\{ \frac{\alpha-k}{2} - 1 \right\} \sin(k+1)\theta \right] d\Psi_\mu \mathbf{e}^1 \\
&\quad + \int_{\mathbb{S}^1} \left[\left\{ \frac{\alpha+k}{2} - 1 \right\} \cos(k-1)\theta - \left\{ \frac{\alpha-k}{2} - 1 \right\} \cos(k+1)\theta \right] d\Psi_\mu \mathbf{e}^2 \\
&= \left\{ \frac{\alpha+k-2}{2} \right\} \int_{\mathbb{S}^1} \begin{pmatrix} \sin(k-1)\theta \\ \cos(k-1)\theta \end{pmatrix} d\Psi_\mu \\
&\quad + \left\{ \frac{\alpha-k-2}{2} \right\} \int_{\mathbb{S}^1} \begin{pmatrix} \sin(k+1)\theta \\ -\cos(k+1)\theta \end{pmatrix} d\Psi_\mu \\
&= \int_{\mathbb{S}^1} \left\{ \left(\frac{\alpha+k-2}{2} \right) i e^{-i(k-1)\theta} - \left(\frac{\alpha-k-2}{2} \right) i e^{i(k+1)\theta} \right\} d\Psi_\mu \\
&= i \left(\frac{\alpha+k-2}{2} \right) \overline{\widehat{\Psi}_\mu(k-1)} - i \left(\frac{\alpha-k-2}{2} \right) \widehat{\Psi}_\mu(k+1).
\end{aligned}$$

Altogether we have

$$\begin{aligned}
(2.1) \quad \mathbf{Z}_{\Omega_1^k}^\alpha(c_J; \mu) &= \left(\frac{\alpha+k-2}{2} \right) \overline{\widehat{\Psi}_\mu(k-1)} + \left(\frac{\alpha-k-2}{2} \right) \widehat{\Psi}_\mu(k+1); \\
\mathbf{Z}_{\Omega_2^k}^\alpha(c_J; \mu) &= i \left[\left(\frac{\alpha+k-2}{2} \right) \overline{\widehat{\Psi}_\mu(k-1)} - \left(\frac{\alpha-k-2}{2} \right) \widehat{\Psi}_\mu(k+1) \right].
\end{aligned}$$

Thus $\det \begin{bmatrix} \mathbf{Z}_{\Omega_1^k}^\alpha(c_J; \mu) \\ \mathbf{Z}_{\Omega_2^k}^\alpha(c_J; \mu) \end{bmatrix}$ is the imaginary part of $\mathbf{Z}_{\Omega_1^k}^\alpha(c_J; \mu) \overline{\mathbf{Z}_{\Omega_2^k}^\alpha(c_J; \mu)}$, which is -1 times the real part of

$$\begin{aligned} & \left\{ \left(\frac{\alpha + k - 2}{2} \right) \overline{\widehat{\Psi}_\mu(k-1)} + \left(\frac{\alpha - k - 2}{2} \right) \widehat{\Psi}_\mu(k+1) \right\} \\ & \times \left\{ \left(\frac{\alpha + k - 2}{2} \right) \widehat{\Psi}_\mu(k-1) - \left(\frac{\alpha - k - 2}{2} \right) \overline{\widehat{\Psi}_\mu(k+1)} \right\} \\ & = \left(\frac{\alpha + k - 2}{2} \right)^2 \left| \widehat{\Psi}_\mu(k-1) \right|^2 - \left(\frac{\alpha - k - 2}{2} \right)^2 \left| \widehat{\Psi}_\mu(k+1) \right|^2 \\ & + \operatorname{Re} \left[\left(\frac{\alpha + k - 2}{2} \right) \left(\frac{\alpha - k - 2}{2} \right) \left(\widehat{\Psi}_\mu(k+1) \widehat{\Psi}_\mu(k-1) - \overline{\widehat{\Psi}_\mu(k-1) \widehat{\Psi}_\mu(k+1)} \right) \right] \\ & = \left(\frac{\alpha + k - 2}{2} \right)^2 \left| \widehat{\Psi}_\mu(k-1) \right|^2 - \left(\frac{\alpha - k - 2}{2} \right)^2 \left| \widehat{\Psi}_\mu(k+1) \right|^2, \end{aligned}$$

since $\widehat{\Psi}_\mu(k+1) \widehat{\Psi}_\mu(k-1) - \overline{\widehat{\Psi}_\mu(k-1) \widehat{\Psi}_\mu(k+1)}$ is pure imaginary. We conclude that

$$(2.13) \quad \det \begin{bmatrix} \mathbf{Z}_{\Omega_1^k}^\alpha(c_J; \mu) \\ \mathbf{Z}_{\Omega_2^k}^\alpha(c_J; \mu) \end{bmatrix} = 0 \iff \left| \widehat{\Psi}_\mu(k+1) \right| = \left| \frac{\alpha + k - 2}{\alpha - k - 2} \right| \left| \widehat{\Psi}_\mu(k-1) \right|, \quad \text{all } k.$$

We also have that $\det \begin{bmatrix} \mathbf{Z}_{\Omega_1^k}^\alpha(c_J; \mu) \\ \mathbf{Z}_{\Omega_1^\ell}^\alpha(c_J; \mu) \end{bmatrix}$ is the imaginary part of $\mathbf{Z}_{\Omega_1^k}^\alpha(c_J; \mu) \overline{\mathbf{Z}_{\Omega_1^\ell}^\alpha(c_J; \mu)}$, i.e. the imaginary part of

$$\begin{aligned} & \left\{ \left(\frac{\alpha + k - 2}{2} \right) \overline{\widehat{\Psi}_\mu(k-1)} + \left(\frac{\alpha - k - 2}{2} \right) \widehat{\Psi}_\mu(k+1) \right\} \\ & \times \left\{ \left(\frac{\alpha + \ell - 2}{2} \right) \widehat{\Psi}_\mu(k+1) + \left(\frac{\alpha - \ell - 2}{2} \right) \overline{\widehat{\Psi}_\mu(k+3)} \right\}. \end{aligned}$$

If we now suppose that $\widehat{\Psi}_\mu(n)$ is real for all n , then $\mathbf{Z}_{\Omega_1^k}^\alpha(c_J; \mu)$ is real for all k , and it follows that

$$(2.14) \quad \det \begin{bmatrix} \mathbf{Z}_{\Omega_1^k}^\alpha(c_J; \mu) \\ \mathbf{Z}_{\Omega_1^\ell}^\alpha(c_J; \mu) \end{bmatrix} = \operatorname{Im} \left(\mathbf{Z}_{\Omega_1^k}^\alpha(c_J; \mu) \overline{\mathbf{Z}_{\Omega_1^\ell}^\alpha(c_J; \mu)} \right) = 0, \quad \text{all } k, \ell.$$

We are now ready to construct the measure μ with an appropriate density Ψ_μ . In the case $1 \leq \alpha < 2$ there is a choice of density that is easy to prove positive, and we give that first. Then we give a density for all cases $0 \leq \alpha < 2$, but that is much harder to prove positive. Finally we give a particularly simple proof for the case $\alpha = 0$.

Construction of a density in the case $1 \leq \alpha < 2$:

Define a density $\Psi(\theta)$ by

$$\Psi(\theta) = 1 + 2 \sum_{n=1}^{\infty} b_n \cos(2n\theta) = 1 + \sum_{n=1}^{\infty} b_n \{e^{i2n\theta} + e^{-i2n\theta}\},$$

where

$$\begin{aligned} b_n &= \left| \frac{\alpha + (2n-3)}{\alpha - (2n+1)} \frac{\alpha + (2n-5)}{\alpha - (2n-1)} \cdots \frac{\alpha + 3}{\alpha - 7} \frac{\alpha + 1}{\alpha - 5} \frac{\alpha - 1}{\alpha - 3} \right| \\ &= a_n a_{n-1} \dots a_2 a_1, \quad n \geq 1; \\ \text{with } a_n &= \left| \frac{\alpha + (2n-3)}{\alpha - (2n+1)} \right| = \left| \frac{2n-1-x}{2n-1+x} \right| \text{ if } x = 2 - \alpha. \end{aligned}$$

Then we have

$$\begin{aligned} \widehat{\Psi}(2n) &= b_n = \widehat{\Psi}(-2n), \quad n \geq 1, \\ \widehat{\Psi}(k) &= 0 \text{ if } k \text{ is odd,} \end{aligned}$$

and in particular that $|\widehat{\Psi}(k+1)| = \left| \frac{\alpha+k-2}{\alpha-k-2} \right| |\widehat{\Psi}(k-1)|$ for all $k \geq 1$. Now choose a measure μ giving rise to the density Ψ . In the case $1 \leq \alpha < 2$ we have $\left| \frac{\alpha+k-2}{\alpha-k-2} \right| = -\frac{\alpha+k-2}{\alpha-k-2}$ for $k \geq 1$, and so from (2.12) we actually obtain that $\mathbf{Z}_{\Omega_1^k}^\alpha(c_J; \mu) = 0$ for all $k \geq 1$, and that $\mathbf{Z}_{\Omega_2^k}^\alpha(c_J; \mu)$ is imaginary for all $k \geq 1$. Thus all of the vectors $\left\{ \mathbf{Z}_{\Omega_1^k}^\alpha(c_J; \mu), \mathbf{Z}_{\Omega_2^k}^\alpha(c_J; \mu) \right\}_{k=1}^\infty$ are multiples of the unit vector $(0, 1)$ in the plane (it is the failure of such a conclusion for the case $0 < \alpha < 1$ that forces a different construction below).

We must now show that the density $\Psi(\theta)$ is nonnegative. We have $\Psi(\theta) = \Phi(2\theta)$ where $\widehat{\Phi}(0) = 1$ and

$$\widehat{\Phi}(n) = \widehat{\Phi}(-n) = b_n = a_n a_{n-1} \dots a_2 a_1, \quad n \geq 1.$$

We claim that the nonnegative sequence $\{1, b_1, b_2, \dots\}$ is convex for $0 < x \leq 2$, and has limit 0 as $n \rightarrow \infty$. With this established, the density Φ is a positive sum of Féjer kernels, and hence $\Phi(\theta) \geq 0$. Since $a_n = \frac{2n-1-x}{2n-1+x} = 1 - \frac{2x}{2n-1+x}$ and $\sum_{n=1}^\infty \frac{2x}{2n-1+x} = \infty$, we see that $\lim_{n \rightarrow \infty} b_n = \prod_{n=1}^\infty \left(1 - \frac{2x}{2n-1+x} \right) = 0$. To see the convexity we note that

$$\begin{aligned} b_{n+1} + b_{n-1} - 2b_n &= a_{n+1} a_n [a_{n-1} \dots a_2 a_1] + [a_{n-1} \dots a_2 a_1] - 2a_n [a_{n-1} \dots a_2 a_1] \\ &= [a_{n+1} a_n + 1 - 2a_n] [a_{n-1} \dots a_2 a_1] \end{aligned}$$

is positive if and only if $a_{n+1} a_n + 1 - 2a_n$ is positive. But for $n \geq 2$ and $0 < x \leq 2$, we have $a_n = \frac{2n-1-x}{2n-1+x}$ and so

$$\begin{aligned} a_{n+1} a_n + 1 - 2a_n &= (a_{n+1} - 2) a_n + 1 \\ &= \left(\frac{2n+1-x}{2n+1+x} - 2 \right) \frac{2n-1-x}{2n-1+x} + 1 \\ &= - \left(\frac{2n+1+3x}{2n+1+x} \right) \frac{2n-1-x}{2n-1+x} + 1 \\ &= \frac{(2n+1+x)(2n-1+x) - (2n+1+3x)(2n-1-x)}{(2n+1+x)(2n-1+x)} \\ &= \frac{4x^2 + 4x}{(2n+1+x)(2n-1+x)} > 0. \end{aligned}$$

This calculation is valid also when $n = 1$ and $0 < x \leq 1$, so it remains to consider only the case $n = 1$ and $1 \leq x \leq 2$. But then we have $a_1 = \frac{x-1}{1+x}$ and so

$$\begin{aligned} a_2 a_1 + 1 - 2a_1 &= (a_2 - 2)a_1 + 1 \\ &= \left(\frac{3-x}{3+x} - 2 \right) \frac{x-1}{1+x} + 1 = \frac{6-2x}{3+x} > 0. \end{aligned}$$

Construction of a density in the general case $0 \leq \alpha < 2$:

This time we modify the definition of our density to be

$$\tilde{\Psi}(\theta) = 1 + 2 \sum_{n=1}^{\infty} b_n \cos(2n\theta) = 1 + \sum_{n=1}^{\infty} b_n \{e^{i2n\theta} + e^{-i2n\theta}\},$$

where

$$\begin{aligned} b_n &= \frac{\alpha + (2n-3)}{\alpha - (2n+1)} \frac{\alpha + (2n-5)}{\alpha - (2n-1)} \cdots \frac{\alpha + 3}{\alpha - 7} \frac{\alpha + 1}{\alpha - 5} \frac{\alpha - 1}{\alpha - 3} \\ &= a_n a_{n-1} \cdots a_2 a_1, \quad n \geq 1; \\ \text{where } a_n &= \frac{\alpha + (2n-3)}{\alpha - (2n+1)} = -\frac{2n-1-x}{2n-1+x} \text{ if } x = 2 - \alpha. \end{aligned}$$

Then we have

$$\begin{aligned} \widehat{\Psi}(2n) &= b_n = \widehat{\Psi}(-2n), \quad 1 \leq n \leq N, \\ \widehat{\Psi}(k) &= 0 \text{ if } k \text{ is odd,} \end{aligned}$$

and in particular, if $\tilde{\mu}$ is chosen to give rise to the density $\tilde{\Psi}$, then from (2.12) we obtain that $\mathbf{Z}_{\Omega_2^k}^\alpha(c_J; \tilde{\mu}) = 0$ for all $k \geq 1$, and that $\mathbf{Z}_{\Omega_1^k}^\alpha(c_J; \tilde{\mu})$ is real for all $k \geq 1$. Thus all of the vectors $\left\{ \mathbf{Z}_{\Omega_1^k}^\alpha(c_J; \tilde{\mu}), \mathbf{Z}_{\Omega_2^k}^\alpha(c_J; \tilde{\mu}) \right\}_{k=1}^{\infty}$ are multiples of the unit vector $(1, 0)$ in the plane.

Finally, we must show that the density $\tilde{\Psi}(\theta)$ is positive. Now

$$\widehat{\Psi}(2n) = b_n = a_n a_{n-1} \cdots a_2 a_1,$$

and so by Bôchner's theorem (more precisely Herglotz's theorem in this application - see e.g. Rudin [Rud] for an extension to locally compact abelian groups), it suffices to check that the following matrices are positive semidefinite for $n \geq 2$:

$$\begin{aligned} \mathbf{B}_n &= \begin{bmatrix} \widehat{\Psi}(0) & \widehat{\Psi}(2) & \widehat{\Psi}(4) & \cdots & \widehat{\Psi}(2n) \\ \widehat{\Psi}(2) & \widehat{\Psi}(0) & \widehat{\Psi}(2) & \cdots & \widehat{\Psi}(2n-2) \\ \widehat{\Psi}(4) & \widehat{\Psi}(2) & \widehat{\Psi}(0) & \cdots & \widehat{\Psi}(2n-4) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \widehat{\Psi}(2n) & \widehat{\Psi}(2n-2) & \widehat{\Psi}(2n-4) & \cdots & \widehat{\Psi}(0) \end{bmatrix} \\ &= \begin{bmatrix} 1 & a_1 & a_2 a_1 & \cdots & a_n \cdots a_1 \\ a_1 & 1 & a_1 & \cdots & a_{n-1} \cdots a_1 \\ a_2 a_1 & a_1 & 1 & \cdots & a_{n-2} \cdots a_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n \cdots a_1 & a_{n-1} \cdots a_1 & a_{n-2} \cdots a_1 & \cdots & 1 \end{bmatrix}. \end{aligned}$$

Since $a_n = -\frac{2n-1-x}{2n-1+x}$, the matrix \mathbf{B}_n is

$$(2.15) \quad \mathbf{B}_n(x) = \begin{bmatrix} 1 & -\frac{1-x}{1+x} & \frac{3-x}{3+x} \frac{1-x}{1+x} & \cdots & \cdots & (-1)^{n+1} \frac{(2n-3)-x}{(2n-3)+x} \cdots \frac{3-x}{3+x} \frac{1-x}{1+x} \\ -\frac{1-x}{1+x} & 1 & -\frac{1-x}{1+x} & \cdots & \cdots & \vdots \\ \frac{3-x}{3+x} \frac{1-x}{1+x} & -\frac{1-x}{1+x} & 1 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ \vdots & \vdots & & \ddots & 1 & -\frac{1-x}{1+x} \\ (-1)^{n+1} \frac{(2n-3)-x}{(2n-3)+x} \cdots \frac{3-x}{3+x} \frac{1-x}{1+x} & \cdots & \cdots & \cdots & -\frac{1-x}{1+x} & 1 \end{bmatrix},$$

and a standard reduction in matrix theory shows that it is enough to show that $\det \mathbf{B}_n(x) \geq 0$ for all $n \geq 2$.

In the appendix below, we prove that these determinants satisfy the recursion formula

$$(2.16) \quad \frac{\det \mathbf{B}_{n+1}(x)}{\det \mathbf{B}_n(x)} = 2^{2n} \frac{n!(n-1+x)(n-2+x) \cdots (x)}{[(2n-1+x)(2n-3+x) \cdots (1+x)]^2}, \quad n \geq 1.$$

From this recursion we immediately obtain that for $x > 0$, the determinants $\det \mathbf{B}_n(x)$ and $\det \mathbf{B}_{n+1}(x)$ have the same sign. Then since $\det \mathbf{B}_1(x) = 1$, induction shows that

$$(2.17) \quad \det \mathbf{B}_n(x) > 0 \text{ for all } x > 0, n \geq 1.$$

This completes the proof that the matrices \mathbf{B}_n are positive definite for all $n \geq 1$ and $x > 0$, and hence that the density $\tilde{\Psi}$ is positive. We note that this completes the proof of Lemma 2 for all $0 \leq \alpha < 2$.

Construction of the density in the case $\alpha = 0$:

The case $\alpha = 0$ corresponds to the usual singular integrals in the plane, and for this case there is an especially simple proof of the nonnegativity of the density $\tilde{\Psi}$. We simply note that the density $\tilde{\Psi}$ is nonnegative by taking absolute values inside the sum,

$$\tilde{\Psi}(\theta) = 1 + 2 \sum_{n=1}^{\infty} b_n \cos(2n\theta) \geq 1 - 2 \sum_{n=1}^{\infty} |b_n|,$$

and then calculating that

$$\begin{aligned} |b_n| &= |a_n a_{n-1} \cdots a_2 a_1| \\ &= \frac{(2n-3)(2n-5)}{(2n+1)(2n-1)} \cdots \frac{3 \cdot 1}{7 \cdot 5 \cdot 3} \\ &= \frac{1}{(2n+1)(2n-1)} = \frac{1}{2} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right), \end{aligned}$$

hence

$$\sum_{n=1}^{\infty} |b_n| = \sum_{n=1}^{\infty} \frac{1}{2} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right) = \frac{1}{2}.$$

□

Now we show how to adapt the above proof to prove Corollary 1.

Proof of Corollary 1. First we note that if Ω is sufficiently smooth with vanishing integral on the circle, then it is an absolutely convergent sum of the trig functions $\cos n\theta$ and $\sin n\theta$ for $n \geq 1$. Thus a standard limiting argument extends the above failure of energy reversal to any finite vector of such Ω . Now embed the measure $\tilde{\mu}$ with density $\tilde{\Psi}$ constructed above into Euclidean space \mathbb{R}^n via the embedding $\mathbb{R}^2 \ni (x_1, x_2) \rightarrow (x_1, x_2, x_3, \dots, x_n) \in \mathbb{R}^2 \times \mathbb{R}^{n-2}$. Here we are letting the parameter $x = n - \alpha$ lie in the interval $(0, n]$. Then the above proof shows that for cubes J with center $c_J \in \mathbb{R}^2 \times \{0\}$, the gradients $\mathbf{Z}_\Omega^\alpha(c_J; \tilde{\mu})$ of the kernels Ω have their planar projections parallel to $(1, 0)$, and hence all the gradients $\mathbf{Z}_\Omega^\alpha(c_J; \tilde{\mu})$ are perpendicular to the fixed direction $(0, 1, 0, \dots, 0)$ in \mathbb{R}^n . As a consequence, reversal of energy fails in J for the measure $\tilde{\mu}$, and it remains only to show that the density $\tilde{\Psi}$ is positive. But this is implied by the positivity of $\det \mathbf{B}_n(x)$ for $x \in (0, n]$, which follows from the recursion (2.16) and the fact that $\det \mathbf{B}_1(x) = 1 > 0$. \square

3. APPENDIX

We can rewrite the recursion (2.16) above as

$$(3.1) \quad \frac{\det \mathbf{B}_{n+1}(x)}{\det \mathbf{B}_n(x)} = \frac{\Omega_n^n(x-1)}{[\Omega_n^n(\frac{x-1}{2})]^2}, \quad n \geq 1,$$

where for any positive integer n and real number a we define the combinatorial coefficient

$$\Omega_n^n(a) \equiv \frac{(n+a)(n-1+a) \dots (1+a)}{(n)(n-1) \dots (1)}.$$

We now prove the recursion formula (3.1) using the well known block determinant formula

$$(3.2) \quad \det \begin{bmatrix} \mathbf{B} & \mathbf{c} \\ \mathbf{r} & a \end{bmatrix} = a \det \mathbf{B} - \mathbf{r} [\text{co} \mathbf{B}]^{\text{tr}} \mathbf{c} = \det \mathbf{B} \{a - \mathbf{r} \mathbf{B}^{-1} \mathbf{c}\},$$

where \mathbf{B} is an $n \times n$ matrix and \mathbf{r} and \mathbf{c} are n -dimensional row and column vectors respectively. Here $[\text{co} \mathbf{B}]^{\text{tr}}$ denotes the transposed cofactor matrix of \mathbf{B} and the inverse of \mathbf{B} is given by $\mathbf{B}^{-1} = \frac{1}{\det \mathbf{B}} [\text{co} \mathbf{B}]^{\text{tr}}$. If we apply this with $\mathbf{B} = \mathbf{B}_n(x)$ and $\begin{bmatrix} \mathbf{B} & \mathbf{c} \\ \mathbf{r} & a \end{bmatrix} = \mathbf{B}_{n+1}(x)$ we get

$$(3.3) \quad \begin{aligned} \det \mathbf{B}_{n+1}(x) &= \det \begin{bmatrix} \mathbf{B}_n(x) & \mathbf{c}^n(x) \\ \mathbf{r}_n(x) & 1 \end{bmatrix} \\ &= \det \mathbf{B}_n(x) \left\{ 1 - \mathbf{r}_n(x) \mathbf{B}_n(x)^{-1} \mathbf{c}^n(x) \right\}, \end{aligned}$$

where $\mathbf{r}_n(x)$ denotes the n -dimensional row vector consisting of the first n entries of the bottom row of $\mathbf{B}_{n+1}(x)$, and similarly $\mathbf{c}^n(x)$ denotes the n -dimensional column vector consisting of the first n entries of the rightmost column of $\mathbf{B}_{n+1}(x)$. Note also that $\mathbf{r}_n(x)$ and $\mathbf{c}^n(x)$ are transposes of each other.

Motivated by computer algebra calculations, we **define** the column vector

$$(3.4) \quad \mathbf{v}^n(x) \equiv (-1)^{n-1} \left[(-1)^k \binom{n}{k} \Gamma_k^n \left(\frac{x-1}{2} \right) \right]_{k=0}^{n-1},$$

where

$$\Gamma_k^n(a) \equiv \frac{\Gamma(k+a+1) \Gamma(n-k+a)}{\Gamma(n+a+1) \Gamma(a)} = \frac{(k+a) \dots (a)}{(n+a) \dots (n-k+a)}.$$

Lemma 3. For $n \geq 1$ we have

$$\mathbf{B}_n(x)^{-1} \mathbf{c}^n(x) = \mathbf{v}^n(x) .$$

Proof. It suffices to show the vector identity

$$\mathbf{B}_n(x) \mathbf{v}^n(x) = \mathbf{c}^n(x), \quad n \geq 1,$$

and to prove this we will use the well known fact that an n^{th} order difference of a polynomial of degree less than n vanishes. More specifically the polynomial in question will be

$$P_{n-1}(s) \equiv \frac{\Gamma(n-1+s)}{\Gamma(s)} = (n-1+s) \dots (1+s) s.$$

Indeed,

$$\begin{aligned} \mathbf{v}^n(x) &\equiv \left[(-1)^k \binom{n}{n-1-k} \Gamma_{n-1-k}^n \left(\frac{x-1}{2} \right) \right]_{k=0}^{n-1} \\ &= \left[(-1)^k \binom{n}{k+1} \frac{\Gamma(n-k+\frac{x-1}{2}) \Gamma(1+k+\frac{x-1}{2})}{\Gamma(n+1+\frac{x-1}{2}) \Gamma(\frac{x-1}{2})} \right]_{k=0}^{n-1} \\ &= \left[(-1)^{k-1} \binom{n}{k} \frac{\Gamma(n-k+z) \Gamma(k-1+z)}{\Gamma(n+z) \Gamma(-1+z)} \right]_{k=1}^n, \end{aligned}$$

where

$$z = \frac{x-1}{2} + 1 = \frac{x+1}{2}.$$

Now we use

$$\begin{aligned} &\frac{(x-1)(x-3)(x-5) \dots (x-(2n-1))}{(x+1)(x+3)(x+5) \dots (x+(2n-1))} \\ &= \frac{\left(\frac{x-1}{2}\right) \left(\frac{x-1}{2}-1\right) \left(\frac{x-1}{2}-2\right) \dots \left(\frac{x-1}{2}-(n-1)\right)}{\left(\frac{x-1}{2}+1\right) \left(\frac{x-1}{2}+2\right) \left(\frac{x-1}{2}+3\right) \dots \left(\frac{x-1}{2}+n\right)} \\ &= \frac{\Gamma\left(\frac{x-1}{2}+1\right) \Gamma\left(\frac{x-1}{2}+1\right)}{\Gamma\left(\frac{x-1}{2}-(n-1)\right) \Gamma\left(\frac{x-1}{2}+n+1\right)} \\ &= \frac{\Gamma(z)^2}{\Gamma(z-n) \Gamma(z+n)}, \end{aligned}$$

to obtain that

$$\mathbf{B}_n(x) = \left[\frac{\Gamma(z)^2}{\Gamma(z-|j-i|) \Gamma(z+|j-i|)} \right]_{i,j=1}^n$$

Thus the first row of $\mathbf{B}_n(x)$ is

$$\begin{aligned} &\left(1 \quad \frac{x-1}{x+1} \quad \frac{x-3}{x+3} \frac{x-1}{x+1} \quad \dots \quad \frac{(x-1)(x-3)(x-5) \dots (x-(2n-1))}{(x+1)(x+3)(x+5) \dots (x+(2n-1))} \right) \\ &= \left(\frac{\Gamma(z)^2}{\Gamma(z) \Gamma(z)} \quad \frac{\Gamma(z)^2}{\Gamma(z-1) \Gamma(z+1)} \quad \frac{\Gamma(z)^2}{\Gamma(z-2) \Gamma(z+3)} \quad \dots \quad \frac{\Gamma(z)^2}{\Gamma(z-(n-1)) \Gamma(z+(n-1))} \right) \\ &= \left[\frac{\Gamma(z)^2}{\Gamma(z-(k-1)) \Gamma(z+(k-1))} \right]_{k=1}^n. \end{aligned}$$

Thus we get

$$\begin{aligned}
& \left[\frac{\Gamma(z)^2}{\Gamma(z-(k-1))\Gamma(z+(k-1))} \right]_{k=1}^n \cdot \mathbf{v}^n(x) \\
&= - \sum_{k=1}^n (-1)^k \binom{n}{k} \frac{\Gamma(n-k+z)\Gamma(k-1+z)}{\Gamma(n+z)\Gamma(-1+z)} \frac{\Gamma(z)^2}{\Gamma(z-(k-1))\Gamma(z+(k-1))} \\
&= - \sum_{k=1}^n (-1)^k \binom{n}{k} \frac{\Gamma(z)^2}{\Gamma(z+n)\Gamma(z-1)} \frac{\Gamma(z-k+n)}{\Gamma(z-k+1)} \\
&= - \frac{\Gamma(z)^2}{\Gamma(z+n)\Gamma(z-1)} \sum_{k=1}^n (-1)^k \binom{n}{k} \{(z-k+n-1) \dots (z-k+1)\} \\
&= - \frac{\Gamma(z)^2}{\Gamma(z+n)\Gamma(z-1)} \sum_{k=1}^n (-1)^k \binom{n}{k} P_z^n(k)
\end{aligned}$$

where $P_z^n(w) = (z-w+n-1) \dots (z-w+1)$ is a polynomial of degree $n-1$. Now recall that if $\Delta f \equiv f(1) - f(0)$ is the unit difference operator at 0, then

$$\Delta^n f = \sum_{k=0}^n (-1)^k \binom{n}{k} f(k)$$

Thus we have

$$\sum_{k=0}^n (-1)^k \binom{n}{k} P_z^n(k) = \Delta^n P_z^n = 0$$

since P_z^n has degree less than n , and so

$$\begin{aligned}
& \left[\frac{\Gamma(z)^2}{\Gamma(z-(k-1))\Gamma(z+(k-1))} \right]_{k=1}^n \cdot \mathbf{v}^n(x) \\
&= \frac{\Gamma(z)^2}{\Gamma(z+n)\Gamma(z-1)} (z+n-1) \dots (z+1) \\
&= \frac{\Gamma(z)^2}{\Gamma(z+1)\Gamma(z-1)}
\end{aligned}$$

which is the first component of $\mathbf{c}^n(x)$ as required. A similar argument proves the equality of the remaining components, and this completes the proof of Lemma 3. \square

Lemma 4. For $n \geq 1$ we have

$$1 - \mathbf{r}_n(x) \cdot \mathbf{v}_n(x) = \frac{\Omega_n^n(x-1)}{[\Omega_n^n(\frac{x-1}{2})]^2}.$$

Proof. Again, this is an application of the fact that an n^{th} order difference of a polynomial of degree less than n vanishes, but a bit more complicated. Recall that

$$\Omega_n^n(a) \equiv \frac{(n+a)(n-1+a) \dots (1+a)}{(n)(n-1) \dots (1)} = \frac{\Gamma(n+1+a)}{\Gamma(1+a)n!},$$

so that we have

$$\begin{aligned} \frac{\Omega_n^n(x-1)}{[\Omega_n^n(\frac{x-1}{2})]^2} &= \frac{n! (n+x-1)(n-1+x-1) \dots (1+x-1)}{(n+\frac{x-1}{2})^2 (n-1+\frac{x-1}{2})^2 \dots (1+\frac{x-1}{2})^2} \\ &= \frac{\Gamma(n+1) \Gamma(n+x) \Gamma(1+\frac{x-1}{2})^2}{\Gamma(x) \Gamma(n+1+\frac{x-1}{2})^2}. \end{aligned}$$

We also have

$$\begin{aligned} \mathbf{v}^n(x) &\equiv \left[(-1)^k \binom{n}{n-1-k} \Gamma_{n-1-k}^n \left(\frac{x-1}{2} \right) \right]_{k=0}^{n-1} \\ &= \left[(-1)^k \binom{n}{k+1} \frac{\Gamma(n-k+\frac{x-1}{2}) \Gamma(1+k+\frac{x-1}{2})}{\Gamma(n+1+\frac{x-1}{2}) \Gamma(\frac{x-1}{2})} \right]_{k=0}^{n-1}, \end{aligned}$$

and from (2.15), we have

$$\begin{aligned} \mathbf{r}_n(x) &= \left((-1)^n \frac{(2n-1)-x}{(2n-1)+x} \dots \frac{3-x}{3+x} \frac{1-x}{1+x} \dots \dots \frac{3-x}{3+x} \frac{1-x}{1+x} - \frac{1-x}{1+x} \right) \\ &= \left[(-1)^{k+1} \frac{(2k+1)-x}{(2k+1)+x} \dots \frac{3-x}{3+x} \frac{1-x}{1+x} \right]_{k=0}^{n-1} \\ &= \left[\frac{x-(2k+1)}{(2k+2)+x-1} \dots \frac{x-3}{4+x-1} \frac{x-1}{2+x-1} \right]_{k=0}^{n-1} \\ &= \left[\frac{-2k+x-1}{(2k+2)+x-1} \dots \frac{-2+x-1}{4+x-1} \frac{x-1}{2+x-1} \right]_{k=0}^{n-1}, \end{aligned}$$

and hence dividing all factors top and bottom by 2, we get

$$\begin{aligned} \mathbf{r}_n(x) &= \left[\frac{-k+\frac{x-1}{2}}{k+1+\frac{x-1}{2}} \dots \frac{-1+\frac{x-1}{2}}{2+\frac{x-1}{2}} \frac{\frac{x-1}{2}}{1+\frac{x-1}{2}} \right]_{k=0}^{n-1} \\ &= \left[\frac{\Gamma(1+\frac{x-1}{2}) \Gamma(1+\frac{x-1}{2})}{\Gamma(-k+\frac{x-1}{2}) \Gamma(k+2+\frac{x-1}{2})} \right]_{k=0}^{n-1} \\ &= \left[\frac{\Gamma(1+\frac{x-1}{2})^2}{\Gamma(-k+\frac{x-1}{2}) \Gamma(k+2+\frac{x-1}{2})} \right]_{k=0}^{n-1}. \end{aligned}$$

Thus our identity to be proved is

$$\begin{aligned} (3.5) \quad &\sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} \frac{\Gamma(1+\frac{x-1}{2})^2}{\Gamma(-k+\frac{x-1}{2}) \Gamma(k+2+\frac{x-1}{2})} \\ &\quad \times \frac{\Gamma(n-k+\frac{x-1}{2}) \Gamma(k+1+\frac{x-1}{2})}{\Gamma(n+1+\frac{x-1}{2}) \Gamma(\frac{x-1}{2})} \\ &= 1 - \frac{\Gamma(n+1) \Gamma(n+x) \Gamma(1+\frac{x-1}{2})^2}{\Gamma(x) \Gamma(n+1+\frac{x-1}{2})^2}. \end{aligned}$$

If we set $z = 1 + \frac{x-1}{2}$ then this identity becomes

$$\begin{aligned} & \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} \frac{\Gamma(z)^2}{\Gamma(-k-1+z)\Gamma(k+1+z)} \frac{\Gamma(n-k-1+z)\Gamma(k+z)}{\Gamma(n+z)\Gamma(-1+z)} \\ &= 1 - \frac{\Gamma(n+1)\Gamma(n-1+2z)\Gamma(z)^2}{\Gamma(-1+2z)\Gamma(n+z)^2}, \end{aligned}$$

and if we replace k by $k-1$ we get

$$\begin{aligned} & \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \frac{\Gamma(z)^2}{\Gamma(-k+z)\Gamma(k+z)} \frac{\Gamma(n-k+z)\Gamma(k-1+z)}{\Gamma(n+z)\Gamma(-1+z)} \\ &= 1 - \frac{\Gamma(n+1)\Gamma(n-1+2z)\Gamma(z)^2}{\Gamma(-1+2z)\Gamma(n+z)^2}. \end{aligned}$$

Note that the term $k=0$ in the sum on the left would be -1 , so that we can subtract 1 from both sides, and then multiply by -1 to get

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\Gamma(z)^2}{\Gamma(-k+z)\Gamma(k+z)} \frac{\Gamma(n-k+z)\Gamma(k-1+z)}{\Gamma(n+z)\Gamma(-1+z)} \\ &= \frac{\Gamma(n+1)\Gamma(n-1+2z)\Gamma(z)^2}{\Gamma(-1+2z)\Gamma(n+z)^2}, \end{aligned}$$

which is equivalent to

$$(3.6) \quad \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\Gamma(z+n-k)\Gamma(z+k-1)}{\Gamma(z-k)\Gamma(z+k)} = \frac{\Gamma(n+1)\Gamma(z-1)\Gamma(2z+n-1)}{\Gamma(z+n)\Gamma(2z-1)}.$$

We now use

$$\frac{\Gamma(s+m+1)}{\Gamma(s)} = (s+m)(s+m-1)\dots(s+1)s$$

to rewrite (3.6) as

$$(3.7) \quad \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{(z+n-k-1)\dots(z-k)}{(z+k-1)} = n! \frac{(2z+n-2)\dots(2z)(2z-1)}{(z+n-1)\dots(z)(z-1)}.$$

Denote the left and right hand sides of (3.7) by $LHS_n(z)$ and $RHS_n(z)$ respectively. Then the left hand side $LHS_n(z)$ of (3.7) is

$$\begin{aligned} LHS_n(z) &= \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{(z+n-k-1)\dots(z+1-k)([z+k-1]-[2k-1])}{\Gamma(z+k-1)} \\ &= \sum_{k=0}^n (-1)^k \binom{n}{k} (z+n-k-1)\dots(z+1-k) \\ &\quad + \sum_{k=0}^n (-1)^{k+1} \binom{n}{k} \frac{(z+n-k-1)\dots(z+1-k)}{(z+k-1)} (2k-1), \end{aligned}$$

where the first sum on the right hand side above vanishes since it is an n^{th} order difference of the polynomial

$$P(w) \equiv (z+n-w-1)\dots(z+1-w)$$

of degree $n - 1$. Thus we have

$$\begin{aligned} LHS_n(z) &= \sum_{k=0}^n (-1)^{k+1} \binom{n}{k} \frac{(z+n-k-1) \dots (z+2-k) ([z+k-1] - [2k-2])}{(z+k-1)} (2k-1) \\ &= \sum_{k=0}^n (-1)^{k+1} \binom{n}{k} (z+n-k-1) \dots (z+2-k) (2k-1) \\ &\quad + \sum_{k=0}^n (-1)^{k+2} \binom{n}{k} \frac{(z+n-k-1) \dots (z+2-k)}{(z+k-1)} (2k-2) (2k-1), \end{aligned}$$

where the first sum on the right hand side above vanishes since it is an n^{th} order difference of the polynomial

$$P(w) \equiv (z+n-w-1) \dots (z+2-w) (2w-1)$$

of degree $n - 1$. Continuing in this way we get

$$LHS_n(z) = \sum_{k=0}^n (-1)^{k+n} \binom{n}{k} \frac{1}{(z+k-1)} (2k-n) \dots (2k-2) (2k-1).$$

Now the right hand side $RHS_n(z)$ of (3.7) is a quotient of a polynomial of degree n by a polynomial of degree $n + 1$, and so has a partial fraction decomposition of the form

$$RHS_n(z) = n! \frac{(2z+n-2) \dots (2z)(2z-1)}{(z+n-1) \dots (z)(z-1)} = \sum_{k=0}^n \frac{A_k}{z+k-1},$$

for uniquely determined coefficients A_0, \dots, A_n . So the proof of (3.7) has been reduced to proving the identity,

$$(3.8) \quad A_k = (-1)^{k+n} \binom{n}{k} (2k-n) \dots (2k-2) (2k-1).$$

Now A_k is the residue of the meromorphic function $RHS_n(z)$ at $z = -(k-1)$, hence using the notation $\widehat{(z+k-1)}$ to indicate that the factor $(z+k-1)$ is *missing*, we get

$$\begin{aligned} A_k &= \text{res}(RHS_n(z); -(k-1)) \\ &= n! \frac{(2z+n-2) \dots (2z)(2z-1)}{(z+n-1) \dots (z+k) \widehat{(z+k-1)} (z+k-2) \dots (z)(z-1)} \Big|_{z=-(k-1)} \\ &= n! \frac{(2[1-k]+n-2) \dots (2[1-k])(2[1-k]-1)}{([1-k]+n-1) \dots ([1-k]+k) \widehat{([1-k]+k-1)} ([1-k]+k-2) \dots ([1-k])([1-k]-1)} \\ &= n! \frac{(-1)^n (2k-n) \dots (2k-2) (2k-1)}{(n-k) \dots (1) \widehat{(0)} (-1)^k (1) \dots (k-1) (k)} \\ &= (-1)^{n-k} \frac{n!}{(n-k)!k!} (2k-n) \dots (2k-2) (2k-1), \end{aligned}$$

which proves (3.8). This completes the proof of Lemma 4. \square

The proof of our claimed recursion (3.1) is now completed by combining Lemmas 3 and 4 with (3.3).

REFERENCES

- [LaWi] LACEY, MICHAEL T., WICK, BRETT D., Two weight inequalities for Riesz transforms: uniformly full dimension weights, *arXiv:1312.6163v2*.
- [Rud] RUDIN, WALTER, Fourier analysis on groups, *Interscience Publ., New York, 1962*.
- [SaShUr] SAWYER, ERIC T., SHEN, CHUN-YEN, URIARTE-TUERO, IGNACIO, A two weight theorem for α -fractional singular integrals with an energy side condition, *arXiv:1302.5093v7*.
- [SaShUr2] SAWYER, ERIC T., SHEN, CHUN-YEN, URIARTE-TUERO, IGNACIO, A two weight theorem for α -fractional singular integrals in higher dimension, *arXiv:1305.5104v7*.
- [SaShUr3] SAWYER, ERIC T., SHEN, CHUN-YEN, URIARTE-TUERO, IGNACIO, A geometric condition, necessity of energy, and two weight boundedness of fractional Riesz transforms, *arXiv:1310.4484v3*.

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